STOCHASTIC CONTROL WITH ROUGH PATHS

JOSCHA DIEHL, PETER FRIZ AND PAUL GASSIAT

ABSTRACT. We study a class of controlled rough differential equations. It is shown that the value function satisfies a HJB type equation; we also establish a form of the Pontryagin maximum principle. Deterministic problems of this type arise in the duality theory for controlled diffusion processes and typically involve anticipating stochastic analysis. We propose a formulation based on rough paths and then obtain a generalization of Roger's duality formula [L. C. G. Rogers, Pathwise Stochastic Optimal Control. SIAM J. Control Optim. 46, 3, 1116-1132, 2007] from discrete to continuous time. We also make the link to old work of [M. H. A. Davis and G. Burstein, A deterministic approach to stochastic optimal control with application to anticipative optimal control. Stochastics and Stochastics Reports, 40:203–256, 1992].

1. Introduction

In classical works [11, 29] Doss and Sussmann studied the link between ordinary and stochastic differential equations (ODEs and SDEs, in the sequel). In the simplest setting, consider a nice vector field σ and a *smooth* path $B : [0,T] \to \mathbb{R}$, one solves the (random) ordinary differential equation

$$\dot{X} = \sigma(X)\dot{B},$$

so that $X_t = e^{\sigma B_t} X_0$, where $e^{\sigma B_t}$ denotes flow, for unit time, along the vector field $\sigma(\cdot) B_t$. The point is that the resulting formula for X_t makes sense for any continuous path B, and in fact the Itô-map $B \mapsto X$ is continuous with respect to $\|\cdot\|_{\infty;[0,T]}$. In particular, one can use this procedure for every (continuous) Brownian path; the so-constructed SDE solution then solves the Stratonovich equation,

$$dX = \sigma(X) \circ dB = \sigma(X)\, dB + \frac{1}{2}(\sigma\sigma')(X)\, dt.$$

When $B = B(\omega) : [0,T] \to \mathbb{R}^d$ is a multidimensional Brownian motion, which we shall assume from here on, this construction fails and indeed the Itô-map is notorious for its lack of (classical) continuity properties. Nonetheless, many approximations, $B^n \to B$, examples of which include the piecewise linear -, mollifier - and Karhunen-Love approximations, have the property that the corresponding (random) ODE solutions, say to $dX^n = b(X^n)\, dt + \sigma(X^n)\, dB^n$ with $\sigma = (\sigma_1, \ldots, \sigma_d)$, convergence to the solution of the Stratonovich equation

$$dX = b(X)\, dt + \sigma(X) \circ dB.$$

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(At least in the case of piecewise linear approximations, this result is known as Wong-Zakai theorem[1]) It was a major step forward, due to T. Lyons [23], to realize that the multidimensional SDE case can also be understood via deterministic differential equations (known as *rough differential equations*); they do, however, require more complicated driving signals (known as *rough paths*) which in the present context are of the form

$$\mathbf{B}(\omega) : [0, T] \to \mathbb{R}^d \oplus so(d),$$

and "contain", in addition to the Brownian sample path, *Lévy stochastic area*, viewed as process with values in $so(d)$. Among the *many* applications of rough paths to stochastic analysis (eg. [14] and the references therein) let us just mention here that (i) *all Wong–Zakai type results* follow from $B^n \to \mathbf{B}$ in a rough path metric (ii) the (rough)pathwise resolution of SDE can handle immediately situations with anticipating randomness in the coefficients; consistency with anticipating SDE in the sense of Ocone–Pardoux [26] was established in [7].

The purpose of this paper is to explore the *interplay of rough paths with control problems*; more specifically, in the context of *controlled* differential equations with usual aim of maximizing a certain payoff function over a fixed time horizon $[0, T]$. As is well known, the dynamic programming principle, in the context of ODEs, leads to Hamilton–Jacobi (HJ) equations for the value function, i.e. non-linear first order partial differential equations, for the value function. Optimal control can be characterized with the aid of the Pontryagin maximum principle (PMP) which essentially is the method of characteristics applied to the HJ equations. As will be discussed in detail in **Section 3** of this paper, all this can be done for general (i.e. deterministic) rough differential equations, say of the form

$$dX = b(X, \mu)\, dt + \sigma(X)\, d\boldsymbol{\eta},$$

where $\boldsymbol{\eta}$ is a rough path and $\mu = (\mu_t)$ a control. The value function then solves a so-called rough viscosity equation; as introduced in [6, 9]; see also the pathwise stochastic control problems proposed by Lions–Souganidis [21], further studied by Buckdahn–Ma [4]. One can, of course, apply this (rough)pathwise to SDEs, just take $\boldsymbol{\eta} = \mathbf{B}(\omega)$; however, the (optimal) control $\mu = \mu_t(\omega)$ will depend anticipatingly on the Brownian driver. Moreover, the ω-wise optimization has (at first glance) little to with the classical stochastic control problem in which one maximizes the expected value, i.e. an average over all ω's, of a payoff function.

Making the link between deterministic and such classical stochastic control problem is the purpose of **Section 4**. In a discrete time setting, Wets [30] first observed that stochastic optimization problems resemble deterministic optimization problems up to the nonanticipativity restriction on the choice of the control policy. The continuous time setting, i.e. studying the link between *controlled* ordinary and *controlled* stochastic differential equations, goes back to Davis–Burstein [8]. The (meta) theorem here is actually a duality of the form

$$(D1) : \sup_{\nu} \mathbb{E}[...] = \mathbb{E}\left[\sup_{\mu}[... + P^*]\right]$$

for a suitable penality P^*, or more generally

$$(D2) : \sup_{\nu} \mathbb{E}[...] = \inf_{P} \mathbb{E}\left[\sup_{\mu}[... + P]\right].$$

where ν denotes an adapted control, μ a possibly anticipating control, the dots $\ldots$ may stand for a payoff such as $g(X_T)$. Note that $(D2)$ has an immediate practical advantage: any choice of P gives

[1]... although, strictly speaking, the multi-dimensional case is due to M. Clark and Stroock–Varadhan.

an upper bound ("duality bound") on the value function and therefore complements lower bounds obtained from picking a particular strategy ν.

In the context of continuous time optimal stopping problems, $(D1)$ was established by Davis–Karatzas [17], $(D2)$ is due to Rogers [28], see also [16], with important applications to the (Monte Carlo) pricing of American options. The extension of $(D2)$, with concrete martingale penality terms, to general control problem in *discrete time only* was carried out by Rogers in [27], see also [3].

While the discrete time setting of Rogers apriori avoids all technicalities related to measurability (e.g. $\sup_\mu$ vs. $ess\sup_\mu$, the meaning of X controlled by anticipating μ^* ...) such things obviously matter in the work of Davis–Burstein. And indeed, the Ocone–Pardoux techniques (notably the representation of anticipating SDE solutions via flow decomposition) play a key role in their work.

Our contribution in **section 4** is then twofold:

(I). We give a "general" duality result in continuous time and see how it can be specialized to yield a version of Roger's duality $(D2)$ for control problems in continous time. Another specialization leads to the Davis–Burstein result, which we review in this context. (Both approaches are then compared explicitly via computations in LQC problems.)

(II) We make the case that *rough path analysis* is ideal to formulate and analyze these problems. Indeed, it allows to write down, in a meaningful and direct way, all the quantities that one wants to write down - without any headache related to afore-mentioned (measurability/anticipativity) technicalities : throughout, all quantities depend continuously on some abstract rough path $\boldsymbol{\eta}$ and are then - trivially - measurable upon substituion $\boldsymbol{\eta} \leftarrow \mathbf{B}(\omega)$.

2. Notation

For $\alpha > \frac{1}{3}$ denote by $\mathcal{C}^{0,\alpha} = \mathcal{C}^{0,\alpha}(E)$ the space of geometric, α-Hölder rough paths over E, where E is a Banach space chosen according to context. [2] On this space, we denote by $\rho_{\alpha-\text{Höl}}$ the corresponding inhomgenous distances.

Let U be some separable metric space (the control space). Denote by $\mathcal{M}$ the class of measurable controls $\mu : [0,T] \to U$.

When working on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{A}$ will denote the class of progressively measurable controls $\nu : \Omega \times [0,T] \to U$.

3. Deterministic control with rough paths

Let $\eta : [0,T] \to \mathbb{R}^d$ be a smooth path. Write $X = X^{t,x,\mu}$ for the solution to the controlled ordinary differential equation

$$dX_s^{t,x,\mu} = b\left(X_s^{t,x,\mu}, \mu_s\right) ds + \sigma\left(X_s^{t,x,\mu}\right) d\eta_s, \qquad X_t^{t,x,\mu} = x \tag{3.1}$$

Classical control theory allows to maximize $\int_t^T f\left(s, X_s^{t,x,u}, \mu_s\right) ds + g\left(X_T^\mu\right)$ over a class of admissible controls μ. As is well-known,

$$v(t,x) := \sup_\mu \left\{ \int_t^T f\left(s, X_s^{t,x,\mu}, \mu_s\right) ds + g\left(X_T^{t,x,\mu}\right) \right\} \tag{3.2}$$

[2] For $\alpha > \frac{1}{2}$ these are just Hölder continuous E-valued paths. For $\alpha \in (1/3, 1/2]$ additional "area" information is necessary. We refer to [24] and [14] for background on rough path theory.

is the (under some technical conditions: unique, bounded uniformly continuous) viscosity solution to the HJB equation

$$\begin{aligned}-\partial_t v - H\left(t,x,Dv\right) - \left\langle \sigma\left(x\right),Dv\right\rangle \dot{\eta} &= 0,\\ v(T,x) &= g(x).\end{aligned}$$

where H acting on v is given by

$$H\left(t,x,p\right) = \sup_u \left\{ \left\langle b\left(x,u\right),p\right\rangle + f\left(t,x,u\right)\right\} \tag{3.3}$$

Now, (3.1) also makes sense for a driving rough path (including a controlled drift terms to the standard setting of RDEs is fairly straight-forward; for the reader's convenience proofs are given in the appendix). This allows to consider the optimization problem (3.2) for controlled RDEs.

3.1. **HJB equation.** The main result here is that the corresponding value function satisfies a "rough PDE". Such equations go back to Lions-Souganidis (in [21] they consider a pathwise stochastic control problem and give an associated stochastic HJB equation, see also [4]; these correspond to $\boldsymbol{\eta} = \mathbf{B}(\omega)$ in the present section). However their (non-rough) pathwise setup is restricted to commuting diffusion vector fields $\sigma_1,\ldots,\sigma_d$ (actually, [21] consider constant vector fields). Extensions to more general vector fields via a rough pathwise approach was then obtained in [6] (see also [10]).

Definition 1. *Let $\boldsymbol{\eta} \in \mathcal{C}^{0,\alpha}$ be a geometric rough path, $\alpha \in (0,1]$. Assume F, G, ϕ to be such that for every smooth path η there exists a unique BUC viscosity solution to*

$$\begin{aligned}-\partial_t v^\eta - F(t,x,v^\eta,Dv^\eta,D^2 v^\eta) - G(t,x,v^\eta,Dv^\eta)\dot{\eta}_t &= 0,\\ v^\eta(T,x) &= \phi(x).\end{aligned}$$

We say that $v \in BUC$ solves the rough partial differential equation

$$\begin{aligned}-dv - F(t,x,v,Dv,D^2 v)dt - G(t,x,v,Dv)d\boldsymbol{\eta}_t &= 0,\\ v(T,x) &= \phi(x),\end{aligned}$$

if for every sequence of smooth paths η^n such that $\eta^n \to \boldsymbol{\eta}$ in rough path metric we have locally uniformly

$$v^{\eta^n} \to v.$$

Remark 2.
(1) *We remark that uniqueness of a solution, if it exists, is built into the definition (by demanding uniqueness for the approximating problems).*
(2) *In special cases (in particular the gradient noise case of the following theorem) it is possible to define the solution to a rough PDE through a coordinate transformation, if the vector fields in front of the rough path are smooth enough. This approach is followed in* [20].
The two definitions are equivalent, if the coefficients admit enough regularity (see [6]*). In the following theorem the coordinate transformation is not applicable, since σ is only assumed to be* Lip^γ *instead of* $\mathrm{Lip}^{\gamma+2}$.

Theorem 3. *Let $\boldsymbol{\eta} \in \mathcal{C}^{0,\alpha}(\mathbb{R}^d)$ be a rough path, $\alpha \in (\frac{1}{3},\frac{1}{2}]$. Let $\gamma > 1/\alpha$. Let $b : \mathbb{R}^e \times U \to \mathbb{R}^e$ be continuous and let $b(\cdot,u) \in \mathrm{Lip}^1(\mathbb{R}^e)$ uniformly in $u \in U$. Let $\sigma_1,\ldots,\sigma_d \in \mathrm{Lip}^\gamma(\mathbb{R}^e)$. Let $g \in BUC(\mathbb{R}^e)$. Let $f : [0,T]\times\mathbb{R}^e\times U \to \mathbb{R}$ be bounded, continuous and locally uniformly continuous in t,x, uniformly in u.*

For $\mu \in \mathcal{M}$ consider the RDE with controlled drift [3] *(Theorem 26),*

$$dX^{t,x,\mu,\boldsymbol{\eta}} = b\left(X^{t,x,\mu,\boldsymbol{\eta}}, \mu\right) dt + \sigma\left(X^{t,x,\mu,\boldsymbol{\eta}}\right) d\boldsymbol{\eta}, \quad X_t^{t,x,\mu,\boldsymbol{\eta}} = x. \tag{3.4}$$

Then

$$v(t,x) := v^{\boldsymbol{\eta}}(t,x) := \sup_{\mu \in \mathcal{M}} \left\{ \int_t^T f\left(s, X_s^{t,x,\mu,\boldsymbol{\eta}}, \mu_s\right) ds + g\left(X_T^{t,x,\mu,\boldsymbol{\eta}}\right) \right\}$$

is the unique bounded, uniformly continuous viscosity solution to the rough HJ equation

$$\begin{aligned} -dv - H(x, Dv)\, dt - \langle \sigma(x), Dv \rangle\, d\boldsymbol{\eta} = 0, \\ v(T,x) = g(x). \end{aligned} \tag{3.5}$$

Proof. The case $f = 0, \sigma \in \mathrm{Lip}^{\gamma+2}$ appears in [9]. The general case presented here is different, since we cannot use a coordinate transformation. Let a smooth sequence η^n be given, such that $\eta^n \to \boldsymbol{\eta}$ in $\mathcal{C}^{0,\alpha}$. Let

$$v^n(t,x) := \sup_{\mu \in \mathcal{M}} \Xi_{t,x}[\eta^n, \mu],$$

where $\Xi_{t,x}[\boldsymbol{\gamma}, \mu] := \int_t^T f\left(s, X_s^{t,x,\mu,\boldsymbol{\gamma}}, \mu_s\right) ds + g\left(X_T^{t,x,\mu,\boldsymbol{\gamma}}\right)$ for any (rough) path $\boldsymbol{\gamma}$. By classical control theory (e.g. Corollary III.3.6 in [1]) we have that v^n is the unique bounded, continuous viscosity solution to

$$\begin{aligned} -dv^n - H(x, Dv^n)\, dt - \langle \sigma(x), Dv^n \rangle\, d\eta^n = 0, \\ v^n(T,x) = g(x). \end{aligned}$$

Then

$$|v^n(t,x) - v(t,x)| \le \sup_{\mu \in \mathcal{M}} \left| \Xi_{t,x}[\boldsymbol{\eta}, \mu] - \Xi_{t,x}[\eta^n, \mu] \right|.$$

Note that Ξ is continuous in $\boldsymbol{\gamma}$ uniformly in μ (and (t,x)) by Theorem 26. Therefore, v^n converges locally uniformly to v and then, by Definition 1, v solves (3.5). □

Example 4. *In the case with additive noise ($\sigma(x) \equiv Id$) and state-independent gains / drift ($f(s,x,u) = f(s,u), b(x,u) = b(s,u)$), this rough deterministic control problem admits a simple solution. Indeed, if v^0 is the value function to the standard deterministic problem for $\boldsymbol{\eta} \equiv 0$, i.e.*

$$v^0(t,x) = \sup_{\mu \in \mathcal{M}} \left\{ \int_t^T f(s, \mu_s) ds + g\left(x + \int_t^T b(s, \mu_s) ds \right) \right\},$$

then one has immediately (since η only appears in the terminal gain)

$$v^{\boldsymbol{\eta}}(t,x) = v^0(t, x + \eta_T - \eta_t).$$

When v^0 has a nice form, this gives simple explicit solutions. For instance, assuming in addition $f \equiv 0$, U convex and $b(s,u) = u$, v^0 is reduced to a static optimization problem and

$$v^{\boldsymbol{\eta}}(t,x) = \sup_{u \in U} g\big(x + \eta_T - \eta_t + (T-t)u\big).$$

[3]Extension to time-dependent b, σ would be straightforward, see [14, ch. 12]. It is also possible to consider the controlled hybrid RDE/SDE $dX = b(X,\mu)dt + \tilde{\sigma}(X,\mu)dW + \sigma(X)d\boldsymbol{\eta}$, see [9].

3.2. **Pontryagin maximum principle.** If η is smooth, then Theorem 3.2.1 in [31] gives the following optimality criterium.

Theorem 5. *Let η be a smooth path. Assume b, f, g be C^1 in x, such that the derivative is Lipschitz in x, u and bounded and let σ, g be C^1 with bounded, Lipschitz first derivative. Let $\bar{X}, \bar{\mu}$ be an optimal pair for problem* (3.2) *with $t = 0$. Let p be the unique solution to the backward ODE*

$$\begin{aligned} -\dot{p}(t) &= Db(\bar{X}_t, \bar{\mu}_t)p(t) + D\sigma(\bar{X}_t)\dot{\eta}_t p(t) + Df(\bar{X}_t, \bar{\mu}_t), \\ p(T) &= Dg(\bar{X}_T). \end{aligned}$$

Then

$$b(\bar{X}_t, \bar{\mu}_t)p(t) + f(\bar{X}_t, \bar{\mu}_t) = \sup_{u \in U} \left[b(\bar{X}_t, u)p(t) + f(\bar{X}_t, u) \right], \qquad a.e.\ t \in [0, T].$$

Let now $\boldsymbol{\eta}$ be rough. We have the following equivalent statement.

Theorem 6. *Assume the same regularity on b, f, g as in Theorem 5. Assume $\sigma_1, \ldots, \sigma_d \in \mathrm{Lip}^{\gamma+2}(\mathbb{R}^e)$. Let $\boldsymbol{\eta} \in \mathcal{C}^{0,\alpha}$ be a geometric rough path, $\alpha \in (1/3, 1/2]$. Let $\bar{X}, \bar{\mu}$ be an optimal pair. Let p be the unique solution to the controlled, backward RDE*

$$\begin{aligned} -dp(t) &= Db(\bar{X}_t, \bar{\mu}_t)p(t)dt + D\sigma(\bar{X}_t)p(t)d\boldsymbol{\eta}_t + Df(\bar{X}_t, \bar{\mu}_t)dt, \\ p(T) &= Dg(\bar{X}_T). \end{aligned}$$

Then

$$b(\bar{X}_t, \bar{\mu}_t)p(t) + f(\bar{X}_t, \bar{\mu}_t) = \sup_{u \in U} \left[b(\bar{X}_t, u)p(t) + f(\bar{X}_t, u) \right], \qquad a.e.\ t.$$

Remark 7. *This is the necessary condition for an admissible pair to be optimal. In the classical setting there do also exist sufficient conditions (see for example Theorem 3.2.5 in* [31]*). They rely on convexity of the Hamiltonian and therefore will in general not work in our setting because, informally, the $d\boldsymbol{\eta}$-term switches sign all the time.*

We prepare the proof with the following Lemma.

Lemma 8. *Let $\bar{X}, \bar{\mu}$ be an optimal pair. Let μ be any other control. Let $I \subset [0, T]$ be an interval with $|I| = \varepsilon$. Define*

$$\mu^\varepsilon(t) := 1_I(t)\mu(t) + 1_{[0,T]\setminus I}(t)\bar{\mu}(t).$$

Let X^ε be the solution to the controlled RDE (3.4) *corresponding to the control μ^ε. Let Y^ε be the solution to the RDE*

$$Y_t^\varepsilon = \int_0^t Db(\bar{X}_r, \bar{\mu}_r)Y_r^\varepsilon dr + \int_0^t D\sigma(\bar{X}_r)Y_r^\varepsilon d\boldsymbol{\eta}_r + \int_0^t \left[b(\bar{X}_r, \mu_r) - b(\bar{X}_r \bar{\mu}_r) \right] 1_I(r)dr.$$

Then

$$\sup_t |X_t^\varepsilon - \bar{X}_t| = O(\varepsilon), \tag{3.6}$$

$$\sup_t |X_t^\varepsilon - \bar{X}_t - Y_t^\varepsilon| = o(\varepsilon), \tag{3.7}$$

$$\begin{aligned} J(\mu^\varepsilon) - J(\bar{\mu}) &= \langle Dg(\bar{X}_T), Y_T^\varepsilon \rangle \\ &\quad + \int_0^T \left[\langle Df(\bar{X}_r, \bar{\mu}_r), Y_r^\varepsilon \rangle + \left\{ f(\bar{X}_r, \mu_r) - f(\bar{X}_r, \bar{\mu}_r) \right\} 1_I(r) \right] dr + o(\varepsilon). \end{aligned} \tag{3.8}$$

Proof. The joint RDE reads as

$$\begin{aligned}
\bar{X}_t &= x + \int_0^t b\left(\bar{X}_r, \bar{\mu}_r\right) dr + \sigma\left(\bar{X}_r\right) d\boldsymbol{\eta}_r \\
X_t^\varepsilon &= x + \int_0^t b\left(X_r^\varepsilon, \mu_r^\varepsilon\right) dr + \sigma\left(X_r^\varepsilon\right) d\boldsymbol{\eta}_r \\
Y_t^\varepsilon &= \int_0^t Db(\bar{X}_r, \bar{\mu}_r) Y_r^\varepsilon dr + \int_0^t D\sigma(\bar{X}_r) Y_r^\varepsilon d\boldsymbol{\eta}_r + \int_0^t \left[b(\bar{X}_r, \mu_r) - b(\bar{X}_r \bar{\mu}_r)\right] 1_I(r) dr.
\end{aligned}$$

By Theorem 26 we can write the solution $S^\varepsilon = (\bar{X}, X^\varepsilon, Y^\varepsilon)$ as $\phi(t, \tilde{S}_t^\varepsilon)$ where

$$d\phi(t,s) = \hat{\sigma}(\phi(t,s)) d\boldsymbol{\eta}_t$$

and $\tilde{S} = (\tilde{\bar{X}}, \tilde{X}^\varepsilon, \tilde{Y}^\varepsilon)$ with

$$d\tilde{S}_t^\varepsilon = B(t, \tilde{S}_t^\varepsilon, (\bar{\mu}_t, \mu_t^\varepsilon)) dt,$$

where

$$\hat{\sigma}(x_1, x_2, x_3) = \begin{pmatrix} \sigma(x_1) \\ \sigma(x_2) \\ D\sigma(x_1) x_3 \end{pmatrix}$$

and $B(t, (x_1, x_2, x_3), (u_1, u_2))$ is given by

$$\begin{pmatrix} \partial_{x_1} \phi_1^{-1}(t, \phi(t,x)) b(\phi_1(t,x), u_1) \\ \partial_{x_2} \phi_2^{-1}(t, \phi(t,x)) b(\phi_2(t,x), u_2) \\ \partial_{x_3} \phi_3^{-1}(t, \phi(t,x)) \left[Db(\phi_1(t,x), u_1) \phi_3(t,x) + \{ b(\phi_1(t,x), u) - b(\phi_1(t,x), \bar{u}) \} 1_I(t) \right] \\ + \partial_{x_1} \phi_3^{-1}(t, \phi(t,x)) b(\phi_1(t,x), u_1) \end{pmatrix}.$$

From Lemma 3.2.2 in [31] it follows that

$$\sup_t |\tilde{X}_t^\varepsilon - \tilde{\bar{X}}_t| = O(\varepsilon).$$

From this we can deduce (3.6) using the Lispchitzness of ϕ. Furthermore, (3.7) follows from

$$\sup_t |\tilde{X}_t^\varepsilon - \tilde{\bar{X}}_t - \tilde{Y}_t^\varepsilon| = o(\varepsilon)$$

and

$$\phi_1(t, (a, b, b-a)) - \phi_2(t, (a, b, b-a)) - \phi_3(t, (a, b, b-a)) = O(|b-a|^2).$$

(note that $\partial_{x_1} \phi_3^{-1}(t, \phi(t, (x_1, x_2, 0))) = 0$). Finally (3.8) follows by direct calculation from (3.6) and (3.7). □

Proof. We follow the idea of the proof of Theorem 3.2.1 in [31]. Fix $x_0 \in \mathbb{R}^e$. Without loss of generality we take $t = 0$. Define for $\mu \in \mathcal{M}$

$$J(\mu) := \int_0^T f(r, X_r^{0, x_0, \mu, \boldsymbol{\eta}}, \mu_r) dr + g(X_T^{0, x_0, \mu, \boldsymbol{\eta}}),$$

(so that $v(0, x_0) = \inf_{\mu \in \mathcal{M}} J(\mu)$).

Since $\boldsymbol{\eta}$ is geometric, we have

$$\begin{aligned}\langle Dg(\bar{X}_T), Y^\varepsilon_T\rangle &= \langle p_T, Y^\varepsilon_T\rangle - \langle p_0, Y^\varepsilon_0\rangle \\ &= -\int_0^T \langle Df(\bar{X}_r,\bar{\mu}_r), Y^\varepsilon_r\rangle dr + \int_0^T \langle p(r), \left[b(\bar{X}_r,\mu_r) - b(\bar{X}_r,\bar{\mu}_r)\right] 1_I(r)\rangle dr.\end{aligned}$$

Here, Y^ε and I are given as in Lemma 8.

Let any $u \in U$ be given. Let $\mu(t) \equiv u$. Let $t \in [0,T)$ and let $\varepsilon > 0$ small enough such that $I_\varepsilon := [t, t+\varepsilon] \subset [0,T]$. Then, combined with Lemma 8 we get

$$\begin{aligned}0 &\geq J(\mu^\varepsilon) - J(\bar{\mu}) \\ &= \langle Dg(\bar{X}_T), Y^\varepsilon_T\rangle + \int_0^T \left[\langle Df(\bar{x}_r,\bar{\mu}_r), Y^\varepsilon_r\rangle + \left\{f(\bar{X}_r,\mu_r) - f(\bar{X}_r,\bar{\mu}_r)\right\} 1_I(r)\right] dr + o(\varepsilon) \\ &= -\int_0^T \langle Df(\bar{X}_r,\bar{\mu}_r), Y^\varepsilon_r\rangle dr + \int_0^T \langle p(r), \left[b(\bar{X}_r,\mu_r) - b(\bar{X}_r,\bar{\mu}_r)\right] 1_I(r)\rangle dr \\ &\quad + \int_0^T \left[\langle Df(\bar{x}_r,\bar{\mu}_r), Y^\varepsilon_r\rangle + \left\{f(\bar{X}_r,\mu_r) - f(\bar{X}_r,\bar{\mu}_r)\right\} 1_I(r)\right] dr + o(\varepsilon) \\ &= \int_t^{t+\varepsilon} \langle p(r), \left[b(\bar{X}_r,u) - b(\bar{X}_r,\bar{\mu}_r)\right]\rangle + f(\bar{X}_r,u) - f(\bar{X}_r,\bar{\mu}_r) dr + o(\varepsilon).\end{aligned}$$

Dividing by ε and sending $\varepsilon \to 0$ yields, together with the separability of the metric space, the desired result. □

3.3. **Pathwise stochastic control.** We can apply Theorem 3 to enhanced Brownian motion, i.e. take $\boldsymbol{\eta} = \mathbf{B}(\omega)$, Brownian motion enhanced with Lévy's stochastic area which constitutes for a.e. ω a geometric rough path. The (rough)pathwise unique solution to the RDE with controlled drift, $X^{\mu,\boldsymbol{\eta}}|_{\boldsymbol{\eta}=\mathbf{B}(\omega)}$ then becomes a solution to the classical stochastic differential equation (in Stratonovich sense) (Theorem 26).

Proposition 9. *Under the assumptions of Theorem 3, the map*

$$\omega \mapsto \sup_{\mu\in\mathcal{M}} \left\{ \int_t^T f\left(s, X^{\mu,\boldsymbol{\eta}}, \mu_s\right) ds + g\left(X^{\mu,\boldsymbol{\eta}}_T\right)\right\}\Bigg|_{\boldsymbol{\eta}=\mathbf{B}(\omega)}$$

is measurable. In particular, the expected value of the pathwise optimization problem,

$$\bar{v}(t,x) = \mathbb{E}\left[\sup_{\mu\in\mathcal{M}} \left\{ \int_t^T f\left(s, X^{\mu,\boldsymbol{\eta}}, \mu_s\right) ds + g\left(X^{\mu,\boldsymbol{\eta}}_T\right)\right\}\Bigg|_{\boldsymbol{\eta}=\mathbf{B}(\omega)}\right] \tag{3.9}$$

is well-defined.

Proof. The lift into rough pathspace, $\omega \mapsto \mathbf{B}(\omega)$, is measurable. $v^{\boldsymbol{\eta}}$ as element in BUC space depends continuously (and hence: measurably) on the rough path $\boldsymbol{\eta}$. Conclude by composition. □

Remark 10. *Well-definedness of such expressions was a non-trivial technical obstacle in previous works on pathwise stochastic control; e.g.* [8, 4]. *The use of rough path theory allows to bypass this difficulty entirely.*

Remark 11. *Let us explain why we only consider the case where the coefficient $\sigma(x)$ in front of the rough path is not controlled. It would not be too difficult to make sense of RDEs*

$$dX = b(t, X, u)dt + \sigma(X, u)d\boldsymbol{\eta}_t,$$

assuming good regularity for σ and $(u_s)_{s\geq 0}$ chosen in a suitable class (for instance : u piecewise constant, u controlled by $\boldsymbol{\eta}$ in the Gubinelli sense,...). However, in most cases of interest the control problem would degenerate, in the sense that we would have

$$\begin{aligned} v^{\boldsymbol{\eta}}(t,x) &= \sup_{\mu\in\mathcal{M}} \left\{ \int_t^T f\left(s, X_s^{t,x,\mu,\boldsymbol{\eta}}, \mu_s\right) ds + g\left(X_T^{t,x,\mu,\boldsymbol{\eta}}\right) \right\} \\ &= \int_t^T (\sup_{\mu,x} f(s,\mu,x))ds + \sup_x g(x). \end{aligned}$$

The reason is that if σ has enough u-dependence (for instance if $d = 1$, U is the unit ball in $\mathbb{R}^e$ and $\sigma(x,u) = u$) and η has unbounded variation on any interval (as is the case for typical Brownian paths), the system can essentially be driven to reach any point instantly.

In order to obtain nontrivial values for the problem, one would need the admissible control processes to be uniformly bounded in some particular sense (see e.g. [25] *in the Young case, where the (μ_s) need to be bounded in some Hölder space), which is not very natural (for instance, Dynamic Programming and HJB-type pointwise optimizations are then no longer valid).*

4. Duality results for classical stochastic control

We now link the expected value of the pathwise optimization problem, as given in (3.9), to the value function of the (classical) stochastic control problem as exposed in [18, 12],

$$V(t,x) := \sup_{\nu\in\mathcal{A}} \mathbb{E}\left[\int_t^T f\left(s, X_s^{t,x,\nu}, \nu_s\right) ds + g\left(X_T^{t,x,\nu}\right)\right]. \tag{4.1}$$

Here, the sup is taken over all admissible (in particular: adapted) controls ν on $[t,T]$. There are well-known assumptions under which V is a classical (see [18]) resp. viscosity (see [12]) solution to the HJB equation, i.e. the non-linear terminal value problem

$$\begin{aligned} -\partial_t V - F\left(t, x, DV, D^2V\right) &= 0 \\ V(T,\cdot) &= g; \end{aligned}$$

uniqueness holds in suitable classes. In fact, assume the dynamics[4]

$$\begin{aligned} dX_t^{s,x,\nu} &= b\left(X_t^{s,x,\nu}, \nu_t\right) dt + \sum_{i=1}^d \sigma_i\left(X_t^{s,x,\nu}\right) \circ dB_t^i, \qquad X_s^{s,x,\nu} = x, \\ &= \tilde{b}\left(X_t^{s,x,\nu}, \nu_t\right) dt + \sum_{i=1}^d \sigma_i\left(X_t^{s,x,\nu}\right) dB_t^i, \qquad X_s^{s,x,\nu} = x, \end{aligned} \tag{4.2}$$

where $\tilde{b}(x,u) = b(x,u) + \frac{1}{2}\sum_{i=1}^d (\sigma_i \cdot D\sigma_i)(x)$ is the corrected drift. Then the equation is semilinear of the form

$$\begin{aligned} -\partial_t V - \tilde{H}(t,x,DV) - LV = 0, \\ V(T,\cdot) = g. \end{aligned} \tag{4.3}$$

[4] Again, it would be straightforward to include explicit time dependence in the vector fields b, σ_i.

where

$$LV = \frac{1}{2}\,\mathrm{Trace}[(\sigma\sigma^T)D^2V]$$

and $\tilde{H}$ is given by $\tilde{H}(t,x,p) = \sup_u \left\{ \left\langle \tilde{b}(x,u), p \right\rangle + f(t,x,u) \right\}$.

Let us also write

$$L^u V = \tilde{b}(\cdot,u)\,DV + LV, \qquad u \in U.$$

Denote by $\mathcal{A}$ the class of progressively measurable controls $\nu : \Omega \times [0,T] \to U$. As before $\mathcal{M}$ is the class of measurable function $\mu : [0,T] \to U$, with the topology of convergence in measure (with respect to dt).

Theorem 12. *Let $\mathcal{Z}_{\mathcal{F}}$ be the class of all mappings $z : \mathcal{C}^{0,\alpha} \times \mathcal{M} \to \mathbb{R}$ such that*

- *z is bounded, measurable and continuous in $\boldsymbol{\eta} \in \mathcal{C}^{0,\alpha}$ uniformly over $\mu \in \mathcal{M}$*
- *$\mathbb{E}[z(\mathbf{B},\nu)] \geq 0$, if ν is adapted*

Let $b : \mathbb{R}^e \times U \to \mathbb{R}^e$ be continuous and let $b(\cdot,u) \in \mathrm{Lip}^1(\mathbb{R}^e)$ uniformly in $u \in U$. Let $\sigma_1,\dots,\sigma_d \in \mathrm{Lip}^\gamma(\mathbb{R}^e)$, for some $\gamma > 2$, $g \in BUC(\mathbb{R}^e)$ and $f : [0,T] \times \mathbb{R}^e \times U \to \mathbb{R}$ bounded, continuous and locally uniformly continuous in t,x, uniformly in u. Then we have

$$V(t,x) = \inf_{z \in \mathcal{Z}_{\mathcal{F}}} \mathbb{E}\left[\sup_{\mu \in \mathcal{M}} \left\{ \int_t^T f(r, X_r^{t,x,\mu,\boldsymbol{\eta}}, \mu_r)dr + g(X_T^{t,x,\mu,\boldsymbol{\eta}}) + z(\boldsymbol{\eta},\mu) \right\} \Big|_{\boldsymbol{\eta} = \mathbf{B}(\omega)} \right].$$

Where $\mathbf{B}$ denotes the Stratonovich lift of Brownian motion to a geometric rough path and $X^{t,x,\mu,\boldsymbol{\eta}}$ is the solution to the RDE with controlled drift (Theorem 26)

$$dX^{t,x,\mu,\boldsymbol{\eta}} = b\left(X^{t,x,\mu,\boldsymbol{\eta}},\mu\right)dt + \sigma\left(X^{t,x,\mu,\boldsymbol{\eta}}\right)d\boldsymbol{\eta}, \quad X_t^{t,x,\mu,\boldsymbol{\eta}} = x.$$

Remark 13. *Every choice of admissible control $\nu \in \mathcal{A}$ in (4.1) leads to a lower bound on the value function (with equality for $\nu = \nu^*$, the optimal control). In the same spirit, every choice z leads to an upper bound. There is great interest in such duality results, as they help to judge how much room is left for policy improvement. The result is still too general for this purpose and therefore it is an important question, discussed below, to understand whether duality still holds when restricting to some concrete (parametrized) subsets of $\mathcal{Z}_{\mathcal{F}}$.*

Proof. We first note, that the supremum inside the expectation is continuous (and hence measurable) in $\boldsymbol{\eta}$, which follows by the same argument as in the proof of Theorem 3. Since it is also bounded, the expectation is well-defined.

Recall that $X^{t,x,\nu}$ is the solution to the (classical) controlled SDE and that $X^{t,x,\mu,\boldsymbol{\eta}}$ is the solution to the controlled RDE. Let $z \in \mathcal{Z}_{\mathcal{F}}$. Then, using Theorem 26 to justify the step from second to

third line,

$$
\begin{aligned}
V(t,x) &= \sup_{\nu\in\mathcal{A}} \mathbb{E}\left[\int_t^T f(s, X_s^{t,x,\nu}, \nu_s)ds + g(X_T^{t,x,u})\right] \\
&\le \sup_{\nu\in\mathcal{A}} \mathbb{E}\left[\int_t^T f(s, X_s^{t,x,\nu}, \nu_s)ds + g(X_T^{t,x,\nu}) + z(\mathbf{B},\nu)\right] \\
&= \sup_{\nu\in\mathcal{A}} \mathbb{E}\left[\left\{\int_t^T f(s, X_s^{t,x,\mu,\boldsymbol{\eta}}, \mu_s)ds + g(X_T^{t,x,\mu,\boldsymbol{\eta}}) + z(\boldsymbol{\eta},\mu)\right\}_{\mu=\nu,\boldsymbol{\eta}=\mathbf{B}}\right] \\
&\le \mathbb{E}\left[\sup_{\mu\in\mathcal{M}}\left\{\int_t^T f(s, X_s^{t,x,\mu,\boldsymbol{\eta}}, \mu_s)ds + g(X_T^{t,x,\mu,\boldsymbol{\eta}}) + z(\boldsymbol{\eta},\mu)\right\}_{\boldsymbol{\eta}=\mathbf{B}}\right],
\end{aligned}
$$

And to show equality, let

$$
z^*(\boldsymbol{\eta},\mu) := V(t,x) - \int_t^T f(s, X_s^{t,x,\mu,\boldsymbol{\eta}}, \mu_s)ds + g(X_T^{t,x,\mu,\boldsymbol{\eta}}).
$$

Then $z^* \in \mathcal{Z}_{\mathcal{F}}$ and equality is attained. □

4.1. **Example I, inspired by the discrete-time results of Rogers** [27]**.** We now show that Theorem 12 still holds with penalty terms based on martingale increments.

Theorem 14. *Under the (regularity) assumptions of Theorem 12 we have*

$$
V(t,x) = \inf_{h\in C_b^{1,2}} \mathbb{E}\left[\sup_{\mu\in\mathcal{M}}\left\{\int_t^T f\left(s, X^{t,x,\mu,\boldsymbol{\eta}}, \mu_s\right)ds + g\left(X_T^{t,x,\mu,\boldsymbol{\eta}}\right) - M_{t,T}^{t,x,\mu,\boldsymbol{\eta},h}\right\}\Bigg|_{\boldsymbol{\eta}=\mathbf{B}(\omega)}\right],
$$

where

$$
M_{t,T}^{t,x,\mu,\boldsymbol{\eta},h} := h\left(T, X_T^{t,x,\mu,\boldsymbol{\eta}}\right) - h\left(t, X_t^{t,x,\mu,\boldsymbol{\eta}}\right) - \int_t^T \left(\partial_s + L^{\mu_s}\right) h\left(s, X_s^{t,x,\mu,\boldsymbol{\eta}}\right) ds.
$$

That is, Theorem 12 still holds with $\mathcal{Z}_{\mathcal{F}}$ replaced by the set $\{z : z(\boldsymbol{\eta},\mu) = M_{t,T}^{t,x,\mu,\boldsymbol{\eta},h}, h \in C_b^{1,2}\}$. Moreover, if $V \in C_b^{1,2}$ the infimum is achieved at $h^* = V$.

Proof. We have

$$
\begin{aligned}
&V(t,x)\\
&\le \inf_{h\in C_b^{1,2}} \mathbb{E}\left[\sup_{\mu\in\mathcal{M}}\left\{\int_t^T f(s,X_s^{t,x,\mu,\boldsymbol{\eta}},\mu_s)ds + g(X_T^{t,x,\mu,\boldsymbol{\eta}}) - M_{t,T}^{t,x,\mu,\boldsymbol{\eta},h}\right\}\Big|_{\boldsymbol{\eta}=\mathbf{B}}\right]\\
&= \inf_{h\in C_b^{1,2}}\Big(h(t,x)\\
&\qquad + \mathbb{E}\left[\sup_{\mu\in\mathcal{M}}\left\{\int_t^T f(s,X_s^{t,x,\mu,\boldsymbol{\eta}},\mu_s) + (\partial_s + L^{\mu_s})h(s,X_s^{t,x,\mu,\boldsymbol{\eta}})ds + g(X_T^{t,x,\mu,\boldsymbol{\eta}}) - h(T,X_T^{t,x,\mu,\boldsymbol{\eta}})\right\}\Big|_{\boldsymbol{\eta}=\mathbf{B}}\right]\Big)\\
&\le \inf_{h\in C_b^{1,2}}\left(h(t,x) + \int_t^T \sup_{x\in\mathbb{R}^e,u\in U}\left[f(s,x,u) + (\partial_s + L^u)h(s,x)\right]ds + \sup_{x\in\mathbb{R}^e}\left[g(x) - h(T,x)\right]\right)\\
&\le \inf_{h\in S_s^+}\left(h(t,x) + \int_t^T \sup_{x\in\mathbb{R}^e,u\in U}\left[f(s,x,u) + (\partial_s + L^u)h(s,x)\right]ds + \sup_{x\in\mathbb{R}^e}\left[g(x) - h(T,x)\right]\right)\\
&\le \inf_{h\in S_s^+} h(t,x).
\end{aligned}
$$

where the first inequality follows from (the proof of) Theorem 12 and S_s^+ denotes the class of smooth super solutions of the HJB equation.

But in fact the infimum of smooth supersolutions is equal to the viscosity solution V, all inequalities are actually equalities and the result follows. This can be proved via a technique due to Krylov [19] which he called "shaking the coefficients". For the reader's convenience let us recall the argument.

Extending by continuity $\tilde{b}, \sigma$ and f to $t \in (-\infty,\infty)$, define for $\varepsilon > 0$,

$$
F^\varepsilon(t,x,p,X) \quad := \quad \sup_{u\in U,|s|,|e|\le\varepsilon}\left[\langle\tilde{b}(x+e,u),p\rangle + \frac{1}{2}Tr(\sigma\sigma^T(x+e)X) + f(t+s,x+e,u)\right],
$$

and consider V^ε the unique viscosity solution to

$$
\begin{cases} -\frac{\partial V^\varepsilon}{\partial t} - F^\varepsilon(t,x,DV^\varepsilon,D^2V^\varepsilon) &= 0,\\ V^\varepsilon(T,\cdot) &= g.\end{cases}
$$

By (local) uniform continuity of b, σ, f one can actually show that $V \to V^\varepsilon$ as $\varepsilon \to 0$, locally uniformly. This can be done for instance by interpreting V^ε as the value function of a stochastic control problem.

Now take some smoothing kernel ρ_ε with $\int_{\mathbb{R}^{e+1}}\rho_\varepsilon = 1$ and $\mathrm{supp}(\rho_\varepsilon) \subset [-\varepsilon,\varepsilon]^{e+1}$, and define $V_\varepsilon := V^\varepsilon * \rho_\varepsilon$. Clearly by definition of F^ε, for each $|s|,|e| \le \varepsilon$, $V^\varepsilon(\cdot - s, \cdot - e)$ is a supersolution to the HJB equation $-\partial_t V - F(t,x,DV,D^2V) = 0$. Since F is convex in (DV, D^2V) it follows that

$$
V_\varepsilon \quad = \quad \int_{[-\varepsilon,\varepsilon]^{e+1}} V^\varepsilon(\cdot - s, \cdot - e)\rho_\varepsilon(s,e)dsde
$$

is again a (smooth) supersolution (for the details see the appendix in [2]). Finally it only remains to notice that $|V - V_\varepsilon| \le |V - V * \rho_\varepsilon| + |(V - V^\varepsilon) * \rho_\varepsilon| \to 0$ (locally uniformly). $\square$

Remark 15. *Note that*

$$V^h(t,x) := \mathbb{E}\left[\sup_{\mu\in\mathcal{M}} \left\{ \int_t^T f\left(s, X^{t,x,\mu,\boldsymbol{\eta}}, \mu_s\right) ds + g\left(X_T^{t,x,\mu,\boldsymbol{\eta}}\right) - M_{t,T}^{t,x,\mu,\boldsymbol{\eta},h} \right\}\Big|_{\boldsymbol{\eta}=\mathbf{B}(\omega)} \right],$$

for fixed x,t, is precisely of the form 3.9 with f resp. g replaced by $\tilde{f}$ resp. $\tilde{g}$, given by

$$\begin{aligned} \tilde{f}(s,\cdot,\mu) &= f(s,\cdot,\mu) + (\partial_s + L^\mu)\, h(s,\cdot), \\ \tilde{g}(\cdot) &= g(\cdot) + h(T,\cdot) - h(t,x). \end{aligned}$$

The point is that the inner pathwise optimization falls directly into the framework of Section 3.

Remark 16. *For $\boldsymbol{\eta}$ a geometric rough path, we may apply the chain rule to $h(s,X_s)$ and obtain*

$$h(T,X_T) - h(t,x) = \int_t^T \langle Dh(s,X_s), b(s,X_s,\mu_s)ds + \sigma(X_s)d\boldsymbol{\eta}_s\rangle.$$

It follows that the penalization may also be rewritten in a (rough) integral form

$$M_{t,T}^{t,x,\mu,\boldsymbol{\eta},h} = \int_t^T \langle Dh(s,X_s), \sigma(X_s)d\boldsymbol{\eta}_s\rangle + \int_t^T \left\{ \langle (b-\tilde{b})(s,X_s,\mu_s), Dh(s,X_s)\rangle + \mathrm{Tr}[(\sigma\sigma^T)D^2h](s,X_s) \right\} ds.$$

Note that for $\boldsymbol{\eta} = \mathbf{B}$ and adapted ν, this is just the Itô integral $\int_t^T \langle Dh(s,X_s), \sigma(X_s)dB_s\rangle$.

Remark 17. *If one were to try anticipating stochastic calculus, in the spirit* [8]*, to implement Roger's duality in continuous time, then - leaving aside all other technical (measurability) issues that have to be dealt with - more regularity on the coefficient will be required. This is in stark contrast to the usual understanding in SDE theory that rough paths require more regularity than Itô theory.*

Example 18. *From example 4 we can see that in some special cases this method gives explicit upper bounds. Assume :*

- *additive noise ($\sigma \equiv Id$),*
- *state-independent drift $b \equiv b(u)$,*
- *running gain $f(s,x,u) = f^0(u) + \nabla h(x)\cdot b(u)$, with h subharmonic ($\Delta h \geq 0$).*

Then for the penalty corresponding to $h(t,x) = h(x)$, the inner optimization problem is given by

$$\sup_{\mu\in\mathcal{M}} \left\{ \int_t^T (f^0(\mu_s) + \langle \nabla h(X_s^{t,x,\mu,\boldsymbol{\eta}}), b(\mu_s)\rangle) ds + g\left(X_T^{t,x,\mu,\boldsymbol{\eta}}\right) - \int_t^T (\langle \nabla h(X_s^{t,x,\mu,\boldsymbol{\eta}}), b(\mu_s)\rangle + \frac{1}{2}\Delta h(X_s^{t,x,\mu,\boldsymbol{\eta}}))ds \right\}$$

$$\leq \sup_{\mu\in\mathcal{M}} \left\{ \int_t^T f^0(\mu_s)\, ds + (g-h)\left(X_T^{t,x,\mu,\boldsymbol{\eta}}\right) \right\} \quad = \quad V^{0,h}(t, x+\eta_T - \eta_t),$$

where $V^{0,h}$ is the value function to the standard control problem

$$V^0(t,x) = \sup_{\mu\in\mathcal{M}} \left\{ \int_t^T f^0(\mu_s)ds + (g-h)\left(x + \int_t^T \mu_s ds\right) \right\}.$$

From Theorem 14, we then have the upper bound

$$V(t,x) \leq h(x) + \mathbb{E}\left[V^{0,h}(t, x + B_T - B_t)\right].$$

Remark 19. *As in Remark 11, one can wonder how Theorem 14 could translate in the case where σ depends on u. As mentioned in that remark, under reasonable conditions on σ the control problem degenerates so that* for any choice of h*, say for piecewise-constant controls μ, we can expect that*

$$\mathbb{E}\left[\sup_{\mu}\left\{\int_t^T f(s,X_s^{t,x,\mu,\boldsymbol{\eta}},\mu_s)+(\partial_t+L^{\mu_s})h(s,X_s^{t,x,\mu,\boldsymbol{\eta}})ds+g(X_T^{t,x,\mu,\boldsymbol{\eta}})-h(T,X_T^{t,x,\mu,\boldsymbol{\eta}})\right\}\Bigg|_{\boldsymbol{\eta}=\mathbf{B}}\right]\Bigg)$$

$$=\int_t^T \sup_{x\in\mathbb{R}^e,u\in U}\left[f(s,x,u)+(\partial_t+L^u)h(s,x)\right]ds+\sup_{x\in\mathbb{R}^e}\left[g(x)-h(T,x)\right].$$

In other words there is nothing to be gained from considering the (penalized) pathwise optimization problem, as we always get

$$V(t,x)\le h(t,x)+\int_t^T \sup_{x\in\mathbb{R}^e,u\in U}\left[f(s,x,u)+(\partial_t+L^u)h(s,x)\right]ds+\sup_{x\in\mathbb{R}^e}\left[g(x)-h(T,x)\right]$$

which is in fact clear from a direct application of Itô's formula (or viscosity comparison).

4.2. **Example II, inspired by Davis–Burstein** [8]. Under certain concavity assumptions, it turns out that linear penalization is enough.

Theorem 20. *Let g be as in Theorem 12 and assume $f=0$; furthermore make the (stronger) assumption that $b\in C_b^5$, $\sigma\in C_b^5$, $\sigma\sigma^T>0$, and that (4.1) has a feedback solution u^* which is continuous, C^1 in t and C_b^4 in x, taking values in the interior of U. Assume that U is a compact convex subset of $\mathbb{R}^n$,*

Let $Z^{t,x,\boldsymbol{\eta}}$ be the solution starting from x at time t to :

$$dZ \quad = \quad b\left(Z,u^*(t,Z)\right)dt+\sigma\left(Z\right)d\boldsymbol{\eta}-b_u(Z,u^*(t,Z))u^*(t,Z)dt, \tag{4.4}$$

let $W(t,x):=W(t,x;\boldsymbol{\eta}):=g(Z_T^{t,x,\boldsymbol{\eta}})$ and assume that

$$\forall(t,x),\quad u\mapsto\langle b(x,u),DW(t,x;\mathbf{B})\rangle \qquad \text{is strictly concave, a.s.} \tag{4.5}$$

Then

$$V(t,x)=\inf_{\lambda\in A}\mathbb{E}[\sup_{\mu\in\mathcal{M}}\left\{\int_t^T f(r,X_r^{t,x,\mu,\boldsymbol{\eta}},\mu_r)dr+g(X_T^{t,x,\mu,\boldsymbol{\eta}})+\int_t^T\langle\lambda(r,X_r^{t,x,\mu,\boldsymbol{\eta}},\boldsymbol{\eta}),\mu_r\rangle dr\right\}\Bigg|_{\boldsymbol{\eta}=\mathbf{B}(\omega)}].$$

Where A is the class of all $\lambda:[0,T]\times\mathbb{R}^e\times\mathcal{C}^{0,\alpha}\to\mathbb{R}^d$ such that

- *λ is bounded and uniformly continuous on bounded sets*
- *λ is future adapted, i.e. for any fixed t,x, $\lambda(t,x,\mathbf{B})\in\mathcal{F}_{t,T}$*
- *$\mathbb{E}[\lambda(t,x,\mathbf{B})]=0$ for all t,x.*

That is, Theorem 12 still holds with $\mathcal{Z}_{\mathcal{F}}$ replaced by the set

$$\{z:z(\boldsymbol{\eta},\mu)=\int_t^T\langle\lambda(s,X_r^{t,x,\mu,\boldsymbol{\eta}},\boldsymbol{\eta}),\mu_s\rangle ds,\lambda\in A\}.$$

Moreover the infimum is achieved with $\lambda^*(t,x,\boldsymbol{\eta}):=b_u^T(t,u^*(t,x))DW(t,x;\boldsymbol{\eta})$.

Remark 21. *The concavity assumption is difficult to verify for concrete examples. It holds for the linear quadratic case, which we treat in Section 4.3.*

Remark 22. *The case of running cost f is, as usual, easily covered with this formulation. Indeed, let the optimal control problem be given as*

$$dX = b(X,\nu)dt + \sigma(X)\circ dW,$$
$$V(t,x) = \sup_{\nu} \mathbb{E}[\int_t^T f(X,\nu)dr + g(X_T)].$$

Define the new component

$$dX_t^{d+1} = f(X,u)dt, \qquad X_t^{d+1} = x.$$

Then the theorem yields that the penalty

$$\begin{aligned}\lambda^*(t,x) &:= (b_u, f_u)\cdot (D_{x_{1\ldots d}} g(Z_T) + D_{x_{1\ldots d}} Z_T^{e+1}, D_{x_{e+1}} Z_T^{e+1}) \\ &= (b_u, f_u)\cdot (D_{x_{1\ldots d}} g(Z_T) + D_{x_{1\ldots d}} Z_T^{e+1}, 1).\end{aligned}$$

is optimal, where

$$\begin{aligned}dZ &= b(Z,u^*)dt + \sigma(Z)d\boldsymbol{\eta} - b_u(Z,u^*)u^*dt, \\ dZ^{e+1} &= f(Z,u^*)dt - f_u(Z,u^*)u^*dt.\end{aligned}$$

Proof. From (the proof of) Theorem 12 we know $V(t,x) \leq \inf_{\lambda\in A}\mathbb{E}[\ldots]$. The converse direction is proven in [8] by using [5]

$$\lambda^*(t,x,\boldsymbol{\eta}) = b_u^T(t,u^*(t,x))DW(t,x).$$

For the reader's convenience we provide a sketch of the argument below. □

Sketch of the Davis-Burstein argument. We have assumed that the optimal control for the stochastic problem (4.1) is given in feedback form by $u^*(t,x)$. Write $X^{t,x,*} := X^{t,x,u^*}$.

Recall that $Z^{t,x,\boldsymbol{\eta}}$ is the solution starting from x at time t to :

$$dZ \;=\; b\left(Z,u^*(t,Z)\right)dt + \sigma\left(Z\right)d\boldsymbol{\eta} - b_u(Z,u^*(t,Z))u^*(t,Z)dt.$$

Assume that $W(t,x) := W(t,x;\boldsymbol{\eta}) = g(Z_T^{t,x,\boldsymbol{\eta}})$ is a (viscosity) solution to the rough PDE

$$-\partial_t W - \langle b(x,u^*(t,x)) - b_u(x,u^*(t,x))u^*(t,x), DW\rangle - \langle\sigma\left(x\right), DW\rangle\,\dot{\boldsymbol{\eta}} \;=\; 0, \tag{4.6}$$

and assume that W is differentiable in x.

We assumed that

$$\forall (t,x),\quad u\mapsto \langle b(x,u), DW(t,x)\rangle \qquad \text{is strictly concave.}$$

It then follows that

$$\langle b(x,u^*(t,x)) - b_u(x,u^*(t,x))u^*(t,x), DW\rangle \;=\; \sup_{u\in U}\langle b(x,u) - b_u(x,u^*(t,x))u, DW\rangle\,. \tag{4.7}$$

Because of (4.7) the PDE (4.6) may be rewritten as

$$\begin{aligned} & -\partial_t W - \langle b(x,u^*(t,x)), DW\rangle - \langle u^*(t,x), \lambda^*(t,x;\boldsymbol{\eta})\rangle - \langle\sigma\left(x\right), DW\rangle\,\dot{\boldsymbol{\eta}} \\ = \;& -\partial_t W - \sup_{u\in U}\left\{\langle b(x,u), DW\rangle - \langle u, \lambda^*(t,x;\boldsymbol{\eta})\rangle\right\} - \langle\sigma\left(x\right), DW\rangle\,\dot{\boldsymbol{\eta}} \\ = \;& 0.\end{aligned}$$

[5]The paper of Davis–Burstein predates rough path theory and their proof relies on anticipating stochastic calculus.

By verification it follows that actually W is also the value function of the problem with penalty λ^*, and the optimal control is given by u^*, i.e.

$$\begin{aligned} W(t,x) &= W(t,x;\boldsymbol{\eta}) = \sup_{\mu\in\mathcal{M}} \left[g(X_T^{t,x,\mu,\boldsymbol{\eta}}) - \int_t^T \left\langle \lambda^*(s, X_s^{t,x,\mu,\boldsymbol{\eta}};\boldsymbol{\eta}), \mu_s \right\rangle ds \right] \\ &= g(X_T^{t,x,u^*,\boldsymbol{\eta}}) - \int_t^T \left\langle \lambda^*(s, X_s^{t,x,u^*,\boldsymbol{\eta}};\boldsymbol{\eta}), u^*(s, X_s^{t,x,u^*,\boldsymbol{\eta}}) \right\rangle ds \end{aligned}$$

Then, by Theorem 26 we have (if the convexity assumption (4.5) is satisfied a.s. by $\boldsymbol{\eta} = \mathbf{B}(\omega)$)

$$W(t,x;\mathbf{B}) = g(X_T^{t,x,*}) - \int_t^T \left\langle \lambda^*(s, X_s^{t,x,*};\mathbf{B}), u^*(s, X_s^{t,x,*}) \right\rangle ds.$$

It follows in particular that for the original stochastic control problem

$$\begin{aligned} V(t,x) = \sup_{\nu\in\mathcal{A}} \mathbb{E}\left[g(X_T^{t,x,\nu})\right] &= \mathbb{E}\left[g(X_T^{t,x,*})\right] \\ &= \mathbb{E}\left[g(X_T^{t,x,*}) - \int_t^T \left\langle \lambda^*(s, X_s^{t,x,*};\mathbf{B}), u^*(s, X_s^{t,x,*}) \right\rangle ds \right] \\ &= \mathbb{E}\left[\sup_{\mu\in\mathcal{M}} \left\{ g(X_T^{t,x,\mu,\boldsymbol{\eta}}) - \int_t^T \left\langle \lambda^*(s, X_s^{t,x,\mu,\boldsymbol{\eta}};\boldsymbol{\eta}), \mu_s \right\rangle ds \right\} |_{\boldsymbol{\eta}=\mathbf{B}} \right]. \end{aligned}$$

Here we have used that $\lambda^*(t,x,\mathbf{B})$ is future adapted and $\mathbb{E}[\lambda^*(t,x,\mathbf{B})] = 0\ \forall t,x$. which is shown on p. 227 in [8].

□

Remark 23. *The two different penalizations presented above are based on verification arguments for respectively the stochastic HJB equation and the (rough) deterministic HJB equation. One can then also try to devise an approach based on Pontryagin's maximum principles (both stochastic and deterministic). While this is technically possible, the need to use sufficient conditions in the rough PMP means that it can only apply to the very specific case where σ is affine in x, and in consequence we have chosen not to pursue this here.*

4.3. **Explicit computations in LQC problems.** We will compare the two optimal penalizations in the case of a linear quadratic control problem (both for additive and multiplicative noise).

4.3.1. *LQC with additive noise.* The dynamics are given by [6]

$$dX = (MX + N\nu)dt + dB_t \tag{4.8}$$

and the optimization problem is given by

$$V(t,x) = \sup_{\nu\in\mathcal{A}} \mathbb{E}\left[\tfrac{1}{2}\int_t^T (\langle QX_s, X_s\rangle + \langle R\nu_s, \nu_s\rangle)ds + \tfrac{1}{2}\langle GX_T, X_T\rangle \right]. \tag{4.9}$$

This problem admits the explicit solution (see e.g. Section 6.3 in [31])

$$V(t,x) = \frac{1}{2}\langle P(t)x, x\rangle + \frac{1}{2}\int_t^T Tr(P(s))ds, \tag{4.10}$$

[6]This equation admits an obvious pathwise SDE solution (via the ODE satisfied by $X - B$) so that, strictly speaking, there is no need for rough paths here.

where P is the solution to the matrix Riccati equation

$$\begin{aligned} P(T) &= G, \\ P'(t) &= -P(t)M - {}^tMP(t) + PNR^{-1t}NP(t) - Q, \end{aligned}$$

and the optimal control is then given in feedback form by

$$\nu^*(t,x) = -R^{-1t}NP(t)x.$$

Proposition 24. *For this LQ control problem the optimal penalty corresponding to Theorem 20 is given by*

$$z^1(\boldsymbol{\eta},\mu) = -\int_t^T \langle \lambda^2(s;\boldsymbol{\eta}),\mu_s\rangle ds,$$

where

$$\lambda^1(t;\boldsymbol{\eta}) = -{}^tN\int_t^T e^{{}^tM(s-t)}P(s)d\boldsymbol{\eta}_s.$$

The optimal penalty corresponding to Theorem 14 is given by

$$z^2(\boldsymbol{\eta},\mu) = z^1(\boldsymbol{\eta},\mu) + \gamma^R(\boldsymbol{\eta}),$$

where

$$\gamma^R(\boldsymbol{\eta}) = \int_t^T \langle P(s)X_s^0, d\boldsymbol{\eta}_s\rangle - \frac{1}{2}\int_t^T Tr(P(s))ds,$$

X^0 *denoting the solution to the RDE* $dX = MX + d\boldsymbol{\eta}$ *starting at* (t,x)*. 5 In particular, these two penalizations are equal modulo a random constant (not depending on the control) with zero expectation.*

Proof. The formula for z^1 is in fact already computed in [8, sec. 2.4], so that it only remains to do the computation for the Rogers penalization.

It follows from Remark 16 that

$$\begin{aligned} M_{t,T}^{t,x,\mu,\boldsymbol{\eta},V} &= \int_t^T \langle DV(s,X_s), d\boldsymbol{\eta}_s\rangle - \frac{1}{2}\int_t^T Tr(D^2V(s,X_s))ds \\ &= \int_t^T \langle P(s)X_s^\mu, d\boldsymbol{\eta}_s\rangle - \frac{1}{2}\int_t^T Tr(P(s))ds \\ &= \int_t^T \langle P(s)(X_s^0 + \int_0^s e^{M(s-r)}N\mu_r dr), d\boldsymbol{\eta}_s\rangle - \frac{1}{2}\int_t^T Tr(P(s))ds \\ &= \int_t^T \langle \mu_r, ({}^tN\int_r^T e^{{}^tM(s-r)}P(s)d\boldsymbol{\eta}_s)\rangle dr + \int_t^T \langle P(s)X_s^0, d\boldsymbol{\eta}_s\rangle - \frac{1}{2}\int_t^T Tr(P(s))ds. \end{aligned}$$

Hence we see that this penalization can be written as $z^2 = z^1 + \gamma^R(\boldsymbol{\eta})$, where $\gamma^R(\boldsymbol{\eta})$ does not depend on the chosen control. One can check immediately that $\mathbb{E}[\gamma^R(\boldsymbol{\eta})|_{\boldsymbol{\eta}=\mathbf{B}(\omega)}] = 0$.

□

4.3.2. *LQC with multiplicative noise.* Let the dynamics be given by

$$dX = (MX + N\nu)dt + \sum_{i=1}^{n} C_i X \circ dB_t^i \tag{4.11}$$

$$= (\tilde{M}X + N\nu)dt + \sum_{i=1}^{n} C_i X dB_t^i. \tag{4.12}$$

Denote by $X^{t,x,\mu,\boldsymbol{\eta}}$ the solution starting from x at time t to

$$dX_s^{t,x,\mu,\boldsymbol{\eta}} = (MX_s^{t,x,\mu,\boldsymbol{\eta}} + N\mu)dt + \sum_{i=1}^{n} C_i X_s^{t,x,\mu,\boldsymbol{\eta}} d\boldsymbol{\eta}_t^i$$

and by $\Gamma_{t,s}$ the (matrix) solution to the RDE

$$d_s\Gamma_{t,s} = M\Gamma_{t,s}ds + \sum_{i=1}^{n} C_i\Gamma_{t,s}d\boldsymbol{\eta}_s, \quad \Gamma_{t,t} = I$$

Then

$$X_s^{t,x,\mu,\boldsymbol{\eta}} = \Gamma_{t,s}x + \int_t^s \Gamma_{r,s}N\mu_r dr.$$

For simplicity we now take $d = n = 1$: the general case is only notationally more involved.

The optimization problem is given by

$$V(t,x) = \sup_{\nu\in\mathcal{A}} \mathbb{E}\left[\tfrac{1}{2}\int_t^T (QX_s^2 + R\nu_s^2)ds + \tfrac{1}{2}GX_T^2\right]. \tag{4.13}$$

By Section 6.6 in [31] the value function is again given as

$$V(t,x) = \frac{1}{2}P_t x^2$$

and the optimal control as

$$u^*(t,x) = -R^{-1}NP_t x,$$

where

$$\dot{P}_t + 2P_tM + 2P_tC^2 + Q - N^2R^{-1}P_t^2 = 0, \qquad P_T = G. \tag{4.14}$$

We can then compute explicitely the Davis–Burstein and Rogers penalties :

Proposition 25. *For $t \le r \le T$, define*

$$\Theta_r = \int_r^T P_s\Gamma_{r,s}^2(d\boldsymbol{\eta}_s - Cds).$$

Then the optimal penalty corresponding to Theorem 20 is given by

$$z^1(\boldsymbol{\eta},\mu) = -CNx\int_t^T \Theta_s\mu_s ds,$$

while the optimal penalty corresponding to Theorem 14 is given by

$$z^2(\boldsymbol{\eta},\mu) = C\Theta_t x^2 + CNx\int_t^T \bar{\Gamma}_{t,s}\Theta_s\mu_s ds + CN^2\int_t^T\int_t^T \Gamma_{r\wedge s,r\vee s}\Theta_{r\vee s}\mu_r\mu_s drds.$$

Proof. The optimal penalty stemming from Theorem 20 (see also Remark 22) is given by $\int_t^T \lambda^*(r,x)\mu_r dr$, where

$$\lambda^*(r,x) = N\left(GZ_T^1 \partial_x Z_T^1 + \partial_x Z_T^2\right) - NP(r)x,$$

where

$$\begin{aligned} dZ^1 &= MZ^1 ds + CZ^1 d\boldsymbol{\eta}_t, \quad Z_r^1 = x; \\ dZ^2 &= \frac{1}{2}\left(Q - N^2R^{-1}P(s)^2\right)(Z^1)^2 ds, \quad Z_r^2 = 0. \end{aligned}$$

Since $Z_s^1 = \Gamma_{r,s}x$, this is computed to

$$\begin{aligned} \lambda^*(r,x) &= Nx\left(G\Gamma_{r,T}^2 + \int_r^T \left(Q - N^2R^{-1}P(s)^2\right)\Gamma_{r,s}^2 ds - P(r)\right). \\ &= Nx\left(\left[P(s)\Gamma_{r,s}^2\right]_{s=r}^T + \int_r^T \left(-\dot{P}(s) - 2MP(s) - 2C^2P(s)\right)\Gamma_{r,s}^2 ds\right) \\ &= Nx\left(\int_r^T P(s)\Gamma_{r,s}^2(Cd\boldsymbol{\eta}_s - C^2 ds)\right) \\ &= NCx\Theta_r. \end{aligned}$$

For the optimal penalty corresponding to Theorem 14, we apply again Remark 16 to see that the optimal penalty is given by

$$\begin{aligned} M_{t,T}^{t,x,\mu,\boldsymbol{\eta},V} &= \int_t^T \langle DV(s,X_s), CXd\boldsymbol{\eta}_s\rangle - \int_t^T \mathrm{Tr}[C^2X^2D^2V(s,X_s)]ds \\ &= \int_t^T P_rC|X_r^{t,x,\mu,\boldsymbol{\eta}}|^2 d\boldsymbol{\eta}_r - \int_t^T C^2|X_r^{t,x,\mu,\boldsymbol{\eta}}|^2 P_r dr. \end{aligned}$$

It only remains to perform straightforward computations expanding the quadratic terms and applying Fubini's theorem. □

5. Appendix: RDEs with controlled drift

Theorem 26 (RDE with controlled drift)**.** *Let $\alpha \in (1/3, 1/2]$. Let $\boldsymbol{\eta} \in \mathcal{C}^{0,\alpha}$ a geometric α-Hölder rough path. Let $\gamma > \frac{1}{\alpha}$. Let U be the subset of a separable Banach space. Let $b : \mathbb{R}^e \times U \to \mathbb{R}^e$ sucht that $b(\cdot,u) \in \mathrm{Lip}^1(\mathbb{R}^e)$ uniformly in $u \in U$ (i.e.* $\sup_{u\in U} ||b(\cdot,u)||_{\mathrm{Lip}^1(\mathbb{R}^e)} < \infty$*) and such that $u \mapsto b(\cdot,u)$ is measurable. Let $\sigma_1,\dots,\sigma_d \in \mathrm{Lip}^\gamma(\mathbb{R}^e)$. Let $\mu : [0,T] \to U$ be measurable, i.e. $\mu \in \mathcal{M}$.*

(i) Then there exists a unique $Y \in \mathcal{C}^{0,\alpha}$ that solves

$$Y_t = y_0 + \int_0^t b(Y_r,\mu_r)dr + \int_0^t \sigma(Y)d\boldsymbol{\eta}_r.$$

Moreover the mapping

$$(x_0, \boldsymbol{\eta}) \mapsto Y \in \mathcal{C}^{0,\alpha}$$

is locally Lipschitz continuous, uniformly in $\mu \in \mathcal{M}$.

(ii) Assume moreoever that $U \ni u \mapsto b(\cdot, u) \in \mathrm{Lip}^1(\mathbb{R}^e)$ is Lipschitz. If we use the topology of convergence in measure on $\mathcal{M}$ then

$$\begin{aligned} \mathcal{M} \times \mathbb{R}^e \times \mathcal{C}^{0,\alpha} &\to \mathcal{C}^{0,\alpha} \\ (\mu, x_0, \boldsymbol{\eta}) &\mapsto Y, \end{aligned} \tag{5.1}$$

is continuous.

(iii) If $\nu : \Omega \times [0,T] \to U$ is progressively measurable and $\mathbf{B}$ is the Stratonovich rough path lift of a Brownian motion B, then

$$Y|_{\mu=\nu, \boldsymbol{\eta}=\mathbf{B}} = \tilde{Y}, \qquad \mathbb{P}-a.s., \tag{5.2}$$

where $\tilde{Y}$ is the (classical) solution to the controlled SDE

$$\tilde{Y}_t = y_0 + \int_0^t b(\tilde{Y}_r, \nu_r) dr + \int_0^t \sigma(\tilde{Y}) \circ dB_r.$$

(iv) If moreover $\sigma_1, \ldots, \sigma_d \in \mathrm{Lip}^{\gamma+2}(\mathbb{R}^e)$, we can write $Y = \phi(t, \tilde{Y}_t)$ where ϕ is the solution flow to the RDE

$$\phi(t,x) = x + \int_0^t \sigma(\phi(r,x)) d\boldsymbol{\eta}_r,$$

and $\tilde{Y}$ solves the classical ODE

$$\tilde{Y}_t = x_0 + \int_0^t \tilde{b}(r, \tilde{Y}_r, \mu_r) dr,$$

where we define componentwise

$$\tilde{b}(t,x,u)_i = \sum_k \partial_{x_k} \phi_i^{-1}(t, \phi(t,x)) b_k(\phi(t,x), u).$$

Remark 27. *In the last case, i.e. point (iv), we can immediately use results in* [14] *(Theorem 10.53) to also handle linear vector fields.*

Proof. Denote for $\mu \in \mathcal{M}$

$$Z_t^\mu(\cdot) := \int_0^t b(\cdot, \mu_r) dr,$$

which is a well defined Bochner integral in the space $\mathrm{Lip}^1(\mathbb{R}^e)$ (indeed, by assumption on b, $\int_0^t ||b(\cdot, \mu_r)||_{\mathrm{Lip}^1(\mathbb{R}^e)} dr < \infty$). Then $Z^\mu \in C^{1-H\ddot{o}lder}([0,T], \mathrm{Lip}^1(\mathbb{R}^e))$. Indeed

$$\begin{aligned} ||Z_t^\mu - Z_s^\mu||_{\mathrm{Lip}^1(\mathbb{R}^e)} &= || \int_s^t b(\cdot, \mu_r) dr||_{\mathrm{Lip}^1(\mathbb{R}^e)} \\ &\le \int_s^t ||b(\cdot, \mu_r)||_{\mathrm{Lip}^1(\mathbb{R}^e)} dr \\ &\le |t-s| \sup_{u \in \mathcal{U}} ||b(\cdot, u)||_{\mathrm{Lip}^1(\mathbb{R}^e)}, \end{aligned}$$

so

$$||Z^\mu||_{1-H\ddot{o}lder} \le \sup_{u \in \mathcal{U}} ||b(\cdot, u)||_{\mathrm{Lip}^1(\mathbb{R}^e)} < \infty, \tag{5.3}$$

independent of $\mu \in \mathcal{M}$.

By Theorem 29 we get a unique solution to the RDE

$$dY = f(Y)dZ^{\mu} + \sigma(Y)d\boldsymbol{\eta}$$

where $f : \mathbb{R}^e \to L(\mathrm{Lip}^1(\mathbb{R}^e), \mathbb{R}^e)$ is the evaluation operator, i.e. $f(y)V := V(y)$. This gives existence of the controlled RDE as well as continuity in the starting point and in $\boldsymbol{\eta}$. By (5.3), this is independent of $\mu \in \mathcal{M}$ and we hence have shown (i).

Concerning (ii), assume now that $U \ni u \mapsto b(\cdot, u) \in \mathrm{Lip}^1$ is Lipschitz. Using the representation given in the proof of (i) it is sufficient to realize that if $\mu^n \to \mu \in \mathcal{M}$ in measure, then $Z^{\mu^n} \to Z^{\mu}$ in $C^{\beta-H\ddot{o}lder}([0,T], \mathrm{Lip}^1(\mathbb{R}^e))$, for all $\beta < 1$.

Concerning (iii): first of all, we can regard ν as a measurable mapping from $(\Omega, \mathcal{F})$ into the space of all measurable mappings from $[0,T] \to U$ with the topology of convergence in measure. Indeed, if U is a compact subset of a separable Banach space, then this follows from the equivalence of weak and strong measurability for Banach space valued mappings (Pettis Theorem, see Section V.4 in [32]). If U is a general subset of a separable Banach space, then define $\nu^n : \Omega \to \mathcal{M}$ with $\nu^n(\omega)_t := \Phi^n(\nu(\omega)_t)$. Here Φ^n is a (measurable) nearest-neighbor projection on $\{x_1, \dots, x_n\}$, the sequence $(x_k)_{k \geq 0}$ being dense in the Banach space. Then ν^n is taking values in a compact set and hence by the previous case, is measurable as a mapping to $\mathcal{M}$. Finally ν is the pointwise limit of the ν^n and hence also measurable.

Hence $Y|_{\mu=\nu,\boldsymbol{\eta}=\mathbf{B}}$ is measurable, as the concatentation of measurable maps (here we use the joint continuity of RDE solutions in the control and the rough path, i.e. continuity of the mapping (5.1)).

Now, to get the equality (5.2): we can argue as in [13] using the Riemann sum representation of stochastic integral.

(iv) This follows from Theorem 1 in [9] or Theorem 2 in [5]. □

Remark 28. *One can also prove "by hand" existence of a solution, using a fixpoint argument, like the one used in* [15]. *This way one arrives at the same regularity demands on the coefficients. Using the infinite dimensional setting makes it possible to immediately quote existing results on existence, which shortens the proof immensely. We thank Terry Lyons for drawing our attention to this fact.*

In the proof of the previous theorem we needed the following version of Theorem 6.2.1 in [24].

Theorem 29. *Let V, W, Z be some Banach spaces. Let tensor products be endowed with the projective tensor norm.* [7] *Let $\alpha \in (1/3, 1/2]$. and $\boldsymbol{\eta} \in \mathcal{C}^{0,\alpha}(W)$ and $Z \in C^{\beta-H\ddot{o}lder}([0,T], V)$ for some $\beta > 1 - \alpha$. Let $f : Z \to L(V, Z)$ be Lip^1, let $g : Z \to L(W, Z)$ be Lip^{γ}, $\gamma > p$. Then there exists a unique solution $Y \in \mathcal{C}^{0,\alpha}(Z)$ to the RDE*

$$dY = f(Y)dZ + g(Y)d\boldsymbol{\eta},$$

in the sense of Lyons. [8]

Moreover for every $R > 0$ there exists $C = C(R)$ such that

$$\rho_{\alpha-H\ddot{o}l}(Y, \bar{Y}) \leq C||Z - \bar{Z}||_{\beta-H\ddot{o}lder}$$

whenever (Z, X) and $(\bar{Z}, X)$ are two driving paths with $||Z||_{\beta-H\ddot{o}lder;[0,T]}, ||\bar{Z}||_{\beta-H\ddot{o}lder;[0,T]}, ||X||_{\alpha-H\ddot{o}lder;[0,T]} \leq R$.

[7] See [22] p. 18 for more on the choice of tensor norms which, of course, only matter in an infinite dimensional setting.

[8] See e.g. Definition 5.1 in [22].

Proof. Since Z and X have complementary Young regularity (i.e. $\alpha+\beta>1$) the joint rough path λ over (Z,X), where the missing integrals of Z and the cross-integrals of Z and X are defined via Young integration. So we have

$$\lambda_{s,t} = 1 + \begin{pmatrix} Z_{s,t} \\ X_{s,t} \end{pmatrix} + \begin{pmatrix} \int_s^t Z_{s,r} \otimes dZ_r & \int_s^t Z_{s,r} \otimes d\eta_r \\ \int_s^t \eta_{s,r} \otimes dZ_r & \int_s^t \eta_{s,r} \otimes d\eta_r \end{pmatrix}$$

Then, by Theorem 6.2.1 in [24], there exists a unique solution to the RDE

$$dY = h(Y) d\lambda,$$

where $h=(f,g)$.

We calculate how λ depends on Z. For the first level we have of course $||\lambda^{(1)} - \bar{\lambda}^{(1)}||_{\alpha-Hölder} \le ||Z-\bar{Z}||_{\beta-Hölder}$. For the second level we have, by Young's inequality,

$$\begin{aligned} \sup_{s<t} \frac{|\int_s^t Z_{s,r} dZ_r - \int_s^t \bar{Z}_{s,r} d\bar{Z}_r|}{|t-s|^{2\alpha}} &\le \sup_{s<t} \frac{|\int_s^t Z_{s,r} d\left[Z_r - \bar{Z}_r\right]}{|t-s|^{2\alpha}} + \sup_{s<t} \frac{|\int_s^t Z_{s,r} - \bar{Z}_{s,r} d\bar{Z}_r|}{|t-s|^{2\alpha}} \\ &\le C ||Z||_{\beta-Hölder} ||Z-\bar{Z}||_{\beta-Hölder}, \end{aligned}$$

and similarily

$$\sup_{s<t} \frac{|\int_s^t X_{s,r} dZ_r - \int_s^t X_{s,r} d\bar{Z}_r|}{|t-s|^{2\alpha}} \le C ||Z-\bar{Z}||_{\beta-Hölder;[s,t]}.$$

Plugging this into the continuity estimate of Theorem 6.2.1 in [24] we get

$$\rho_{\alpha-\text{Höl}}(Y,\bar{Y}) \le C||Z-\bar{Z}||_{\beta-Hölder}$$

as desired. □

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